The asymptotic behaviors for super-Brownian motion with absorption

Yaping Zhu

Peking University
(Joint work with Professor Zenghu Li)

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Outline

1. Background

2. Problem formulation

3. Main results
   - Survival probability
   - Large deviation
   - Absorbed mass

4. Proofs
   - Proof of Theorem 1
   - Proof of Theorem 4
   - Proof of Theorem 7

Suppose that ψ is a branching mechanism of the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x),$$

where $\alpha = -\psi'(0) \in (0,\infty)$, $\beta \geq 0$ and Π is a σ-finite measure on $(0,\infty)$ satisfying $\int_{(0,\infty)} (x \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

Under $\mathbb{P}^x$, suppose that $B^{-\rho} = \{B_t^{-\rho} : t \geq 0\}$ is a Brownian motion with drift $-\rho$, starting from $x > 0$.

$\mathcal{M}_F(0,\infty)$ ($\mathcal{M}_F(\mathbb{R})$): the space of finite measures on $(0,\infty)$ ($\mathbb{R}$).

- $\mathcal{X}^{-\rho} = \{\mathcal{X}_t^{-\rho} : t \geq 0\}$: a superprocess with branching mechanism ψ and spatial motion $\{B_t^{-\rho} : t \geq 0\}$.
- $X^{-\rho} = \{X_t^{-\rho} : t \geq 0\}$: a superprocess with branching mechanism ψ and spatial motion $\{B_{t\wedge\tau_0}^{-\rho} : t \geq 0\}$.

For all $f \in C_b^+(0,\infty)$, $\mu \in \mathcal{M}_F(0,\infty)$,

$$\mathbb{E}_\mu\left(\mathrm{e}^{-\left\langle f, X_t^{-\rho}\right\rangle}\right) = \mathrm{e}^{-\langle v_f(\cdot,t),\mu\rangle}, \quad t \geq 0,$$

where $v_f(x,t)$ is the unique positive solution to the equation

$$v_f(x,t) = \mathbb{E}^x[f(B_t^{-\rho}); \tau_0 > t] - \mathbb{E}^x\left[\int_0^t \psi(v_f(B_s^{-\rho}, t-s))\mathrm{d}s; \tau_0 > t\right].$$

For $f \in C_0^+(0,\infty)$, the above equation is equivalent to

$$\begin{cases} \frac{\partial}{\partial t} v_f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v_f(x,t) - \rho\frac{\partial}{\partial x} v_f(x,t) - \psi(v_f(x,t)), \\ v_f(0+,t) = 0, \quad v_f(x,0) = f(x), \quad t \geq 0, \quad x > 0. \end{cases} \tag{1}$$

Related results: SBM with absorption

- Watanabe (1968): existence.
- Li and Shiga (1995): immigration structure, the range of immigration process.
- Li (1999): conditional law.
- Kyprianou, Murillo-Salas and Pérez (2012): the right-most point in the support, absorbed mass.

- Engländer (2004): large deviation.
- Kyprianou, Liu, Murillo-Salas and Ren (2012): additive martingale and derivative martingale.
- Kyprianou and Murillo-Salas (2013): L^p-convergence of additive martingale.
- Ren, Song and Zhang (2021): supermum of the support, extremal process.

Outline

Problem formulation

Theorem (Kyprianou, Murillo-Salas and Pérez, 2012)

Define $R_t := \inf\{y > 0 : X_t^{-\rho}(y, \infty) = 0\}$ and denote the extinction time of $X^{-\rho}$ by

$$\zeta := \inf\{t > 0 : ||X_t^{-\rho}|| = 0\}.$$

(1) For all $\rho \geq \sqrt{2\alpha}$ and $x > 0$, $\mathbb{P}_{\delta_x}(\zeta < \infty) = 1$.
(2) Assume that $-\infty < \rho < \sqrt{2\alpha}$. Then for all $x > 0$,

$$\lim_{t\to\infty} \frac{R_t}{t} = \sqrt{2\alpha} - \rho, \quad \text{on } \{\zeta = \infty\} \quad \mathbb{P}_{\delta_x} - \text{a.s.}$$

Problem 1: When $\rho \geq \sqrt{2\alpha}$, what's the decay rate of $\mathbb{P}_{\delta_x}(\zeta > t)$? Consider the behavior of $||X_t^{-\rho}||$ conditioned on $\zeta > t$.

Problem 2: When $\rho < \sqrt{2\alpha}$, what's the decay rate of $\mathbb{P}_{\delta_x}(R_t > \gamma t)$ for $\gamma > \sqrt{2\alpha} - \rho$? Consider the limiting behavior of $X_t^{-\rho}(\gamma t, \infty)$ conditioned on $R_t > \gamma t$.

Remark: Analogue results for branching Brownian motion (BBM) with absorption can be found in Harris et al. (2006, 2007).

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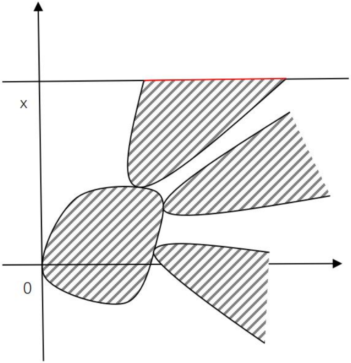

Figure 1. $||X_{D_x}^{-\rho}||$ denote the absorbed mass at x.

Problem formulation

Theorem (Kyprianou, Murillo-Salas and Pérez, 2012)

When $\rho = -\sqrt{2\alpha}$,

$$\mathbb{P}_{\delta_0}\left(||X_{D_x}^{-\rho}|| > t\right) \sim \frac{\sqrt{2\alpha}x e^{\sqrt{2\alpha}x}}{t(\log t)^2}, \quad t \to \infty.$$

Furthermore, when $\rho < -\sqrt{2\alpha}$,

$$\mathbb{P}_{\delta_0}\left(||X_{D_x}^{-\rho}|| > t\right) \sim Ct^{-d}, \quad t \to \infty,$$

where $C = C(x, \rho, \psi)$, $d = \bar{\lambda}_\rho/\lambda_\rho$ and $\lambda_\rho \leq \bar{\lambda}_\rho$ are the two roots of the quadratic equation $\lambda^2 + 2\rho\lambda + 2\alpha = 0$.

Problem 3: For any $\rho > -\sqrt{2\alpha}$, what's the tail behavior of the absorbed mass?

Remark: Maillard (2013), Lalley and Zheng (2015), Berestycki et al. (2017), Corre (2018): the number of absorbed particles in BBM with absorption.

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Theorem (Kyprianou, Murillo-Salas and Pérez, 2012)

When $\rho = -\sqrt{2\alpha}$,

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Assumption

Assume that the branching mechanism ψ satisfies

$$\int^{\infty} \frac{1}{\sqrt{\int_{\lambda^*}^{r} \psi(u)\mathrm{d}u}}\mathrm{d}r < \infty, \tag{2}$$

where λ^* is the largest root of the equation $\psi(\lambda) = 0$.

Remark 1

Let $\mathcal{R}$ be the smallest closed set such that $\mathrm{supp}\mathcal{X}_t^{-\rho} \subseteq \mathcal{R}$, $t \geq 0$. It follows from Sheu (1997) that

$$\mathbb{P}_\mu\left(\mathcal{R} \textit{ is compact}\right) = \mathrm{e}^{-\lambda^*||\mu||},$$

and (2) implies Grey's condition

$$\int^{\infty} \frac{1}{\psi(\lambda)}\mathrm{d}\lambda < \infty.$$

Assumption

Furthermore, we also assume the following integrability condition:

$$\int_{0+} \lambda^{-1}\varphi(\lambda)\mathrm{d}\lambda < \infty,$$

where $\varphi(\lambda) := \lambda^{-1}\psi(\lambda) + \alpha$.

Remark 2

According to Kyprianou, Liu, Murillo-Salas and Ren (2012),

$$\int_{0+} \lambda^{-1}\varphi(\lambda)\mathrm{d}\lambda < \infty \Leftrightarrow \int_{[1,\infty)} \lambda(\log\lambda)\Pi(\mathrm{d}\lambda) < \infty.$$

Outline

Theorem 1 (Survival probability)

For any $\rho > \sqrt{2\alpha}$ and $x > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(\zeta > t)\frac{\sqrt{2\pi t^3}}{xe^{\rho x}}e^{(\frac{1}{2}\rho^2-\alpha)t} = C_1, \tag{3}$$

where C_1 is a positive constant independent of x.

Theorem 2 (Yaglom limit theorem)

For any $\rho > \sqrt{2\alpha}$ and $x > 0$, there exists a probability distribution $\mathbb{Q}$ defined on $(0, \infty)$ such that

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(||X_t^{-\rho}|| \in \cdot|\zeta > t) = \mathbb{Q}(\cdot),$$

and $\mathbb{Q}$ has mean $\frac{2}{C_1\rho^2}$.

Theorem 3 (Weak convergence)

For any $x > 0$ and $r \geq 0$, there exists a probability distribution $\mathbf{Q}_r$ defined on $\mathcal{M}_F(0, \infty)$ such that

$$\mathbb{P}_{\delta_x}(X_t^{-\rho} \in \cdot | \zeta > t + r) \xrightarrow{w} \mathbf{Q}_r(\cdot).$$

(i) When $r = 0$, for any $f \in C_b^+(0, \infty)$,

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}\left(\mathrm{e}^{-\langle f, X_t^{-\rho}\rangle} \middle| \zeta > t\right) = 1 - C_1^{-1} \int_0^\infty 2zf(z)g_f(z)\mathrm{e}^{-\rho z}\mathrm{d}z,$$

where C_1 is defined by (3) and for any $z > 0$,

$$g_f(z) := \mathbb{E}_B^z\left[\exp\left\{-\int_0^\infty \varphi(v_f(Z(r), r))\mathrm{d}r\right\}\right] \in (0, 1],$$

here $\{Z(t)\}_{t\geq 0}$ is a Bessel-3 process starting from z.

Theorem 3 (Weak convergence)

For any $x > 0$ and $r \geq 0$, there exists a probability distribution $\mathbf{Q}_r$ defined on $\mathcal{M}_F(0,\infty)$ such that

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Theorem 3 (Weak convergence)

(ii) For any $r > 0$ and $f \in C_b^+(0,\infty)$, we have

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}\left(\mathrm{e}^{-\langle f, X_t^{-\rho}\rangle} \middle| \zeta > t + r\right)$$
$$= \mathrm{e}^{(\frac{1}{2}\rho^2-\alpha)r} C_1^{-1} \int_0^\infty 2z\mathrm{e}^{-\rho z} \left[(f(z) + u(z,r))\, g_f(z,r) - f(z) g_f(z)\right] \mathrm{d}z,$$

where

$$g_f(z,r) := \mathbb{E}_B^z \left[\exp\left\{-\int_0^\infty \varphi(v_{f+u_r}(Z(s),s))\mathrm{d}s\right\}\right], \quad z > 0.$$

and $u_t(x) := u(x,t) = -\log \mathbb{P}_{\delta_x}(\zeta \le t)$.

Theorem 4 (Large deviation)

For any $\gamma > \sqrt{2\alpha} - \rho$ and $x > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(R_t > \gamma t)\frac{\sqrt{2\pi t}}{1-\mathrm{e}^{-2\gamma x}}\mathrm{e}^{(\frac{1}{2}(\gamma+\rho)^2-\alpha)t}\mathrm{e}^{-(\gamma+\rho)x} = C_2,$$

where C_2 is a positive constant independent of x.

Theorem 5 (Yaglom limit theorem)

For any $\gamma > \sqrt{2\alpha} - \rho$ and $x > 0$, there exists a probability distribution Π such that

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(X_t^{-\rho}(\gamma t, \infty) \in \cdot | R_t > \gamma t) = \Pi(\cdot).$$

Moreover, Π has mean $\frac{1}{C_2(\gamma+\rho)}$.

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For any $\gamma > \sqrt{2\alpha} - \rho$ and $x > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(R_t > \gamma t) \frac{\sqrt{2\pi t}}{1 - \mathrm{e}^{-2\gamma x}} \mathrm{e}^{(\frac{1}{2}(\gamma+\rho)^2 - \alpha)t} \mathrm{e}^{-(\gamma+\rho)x} = C_2,$$

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Theorem 5 (Yaglom limit theorem)

For any $\gamma > \sqrt{2\alpha} - \rho$ and $x > 0$, there exists a probability distribution $\mathbf{\Pi}$ such that

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Moreover, $\mathbf{\Pi}$ has mean $\frac{1}{C_2(\gamma+\rho)}$.

Define $\Theta_{\gamma t}X_t^{-\rho}(\cdot) = X_t^{-\rho}(\cdot + \gamma t)$ and $Y_t^{-\rho} := \Theta_{\gamma t}X_t^{-\rho}$.

Theorem 6 (Weak convergence)

Assume $\gamma > \sqrt{2\alpha} - \rho$. For any $x > 0$ and $s \geq 0$, there exists a probability distribution $\mathbf{\Pi}_s$ on $\mathcal{M}_F(\mathbb{R})$ such that

$$\mathbb{P}_{\delta_x}\left(Y_t^{-\rho} \in \cdot | R_{t+s} > \gamma t\right) \xrightarrow[t\to\infty]{w} \mathbf{\Pi}_s(\cdot).$$

(i) When $s = 0$, for any $f \in C_b^+(0,\infty)$,

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}\left(\mathrm{e}^{-\langle f, Y_t^{-\rho}\rangle} \middle| R_t > \gamma t\right) = 1 - C_2^{-1}\int_0^\infty f(z)h_f(z)\mathrm{e}^{-(\rho+\gamma)z}\mathrm{d}z,$$

where

$$h_f(z) := \mathbb{E}\left(\mathrm{e}^{-\int_0^\infty \varphi(v_f(W_2(r)+z-\gamma r, r))\mathrm{d}r}\right), \quad z > 0,$$

and $\{W_2(t)\}_{t\geq 0}$ is a standard Brownian motion.

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Theorem 6 (Weak convergence)

(ii) For any $s > 0$ and $f \in C_b^+(0,\infty)$, we have

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}\left(\mathrm{e}^{-\langle f, Y_t^{-\rho}\rangle} \middle| R_{t+s} > \gamma t\right) = C_2^{-1}\mathrm{e}^{\frac{1}{2}(\gamma+\rho)^2 s - \alpha s}\int_0^\infty \left[\left(f(z)+u^0(z,s)\right)h_f(z,s) - f(z)h_f(z)\right]\mathrm{e}^{-(\rho+\gamma)z}\mathrm{d}z,$$

and for any $z > 0$,

$$h_f(z,s) = \mathbb{E}\left[\mathrm{e}^{-\int_0^\infty \varphi\left(v_{f+u_s^0}(W_2(r)+z-\gamma r, r)\right)\mathrm{d}r}\right] \in (0,1],$$

where $u_t^0(x) := u^0(x,t) = -\log \mathbb{P}_{\delta_x}(\mathcal{X}_t^{-\rho}(0,\infty) = 0)$.

Absorbed mass

Define

$$u_s(x) := -\log \mathbb{E}_{\delta_0}\left(\mathrm{e}^{-s\|X_{D_x}^{-\rho}\|}\right), \quad x > 0, s \in \mathbb{R},$$

and the singular point

$$s_0 := \inf\{s < 0 : u_s > -\infty\}.$$

Theorem 7 (Tail distribution of the absorbed mass)

When $\rho \geq \sqrt{2\alpha}$, we have $-\infty < s_0 < 0$ and

$$\mathbb{P}_{\delta_0}\left(\|X_{D_x}^{-\rho}\| > t\right) \sim \frac{u'_{s_0}(x)\mathrm{e}^{-u_{s_0}(x)}}{\mathrm{e}^{-s_0 t}\sqrt{\pi t \psi(s_0)}}, \quad t \to \infty.$$

Outline

Theorem 1 (Survival probability)

For any $\rho > \sqrt{2\alpha}$ and $x > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(\zeta > t)\frac{\sqrt{2\pi t^3}}{xe^{\rho x}}e^{(\frac{1}{2}\rho^2-\alpha)t} = C_1,$$

where C_1 is a positive constant independent of x.

Step 1. According to the Laplace transform of $X^{-\rho}$,

$$\mathbb{P}_{\delta_x}(\zeta > t) = 1 - \lim_{\theta\to\infty} \lim_{n\to\infty} e^{-u_n^\theta(x,t)},$$

where $u_n^\theta(x,t)$ is the solution to (1) with boundary condition $u_n^\theta(x,0) = \theta f_n(x)$ and $f_n \in C_0^+(0,\infty)$ satisfying $f_n \uparrow 1_{(0,\infty)}$ as $n \uparrow \infty$.

It follows from monotone convergence theorem that

$$\mathbb{P}_{\delta_x}(\zeta > t) = 1 - \mathrm{e}^{-u(x,t)}.$$

Step 2. Since $\{||\mathcal{X}_t^{-\rho}|| : t \geq 0\}$ is a continuous state branching process (CSBP), naturally we can use the cumulant semigroup of CSBP to give an upper bound for $u(x, t)$.

Lemma

For any $x > 0$,

$$u(x,t) \leq \hat{v}_t, \quad t > 0,$$

where $\hat{v}_t :=\uparrow \lim_{\theta \to \infty} v_t(\theta)$ and $t \mapsto v_t(\theta)$ is the unique positive solution of

$$v_t(\theta) = \theta - \int_0^t \psi(v_s(\theta))\mathrm{d}s, \quad t \geq 0.$$

Moreover, for any $t > 0$, $\hat{v}_t < \infty$ if and only if Grey's condition holds.

Step 3. It follows from Feynman-Kac formula that

$$u(x,t)\frac{\sqrt{2\pi t^3}}{xe^{\rho x}}\mathrm{e}^{(\frac{1}{2}\rho^2-\alpha)(t-1)}$$
$$=\int_{(0,\infty)}\frac{\sqrt{t}}{\sqrt{t-1}}\frac{t}{x}(1-e^{-\frac{2xz}{t-1}})e^{-\rho z}u(z,1)$$
$$\times\mathbb{E}\left[e^{-L_{z,x}(1,t)}|\tau_0^{t-1}(z,x)>t-1\right]e^{-\frac{(x-z)^2}{2(t-1)}}\mathrm{d}z$$
$$\to\int_{(0,\infty)}2ze^{-\rho z}u(z,1)\mathbb{E}_B^z(e^{-\int_1^\infty\varphi(u(Z(r-1),r))\mathrm{d}r})\mathrm{d}z$$

where $L_{z,x}(1,t)=\int_1^t\varphi(u(B_{z,x}^{t-1}(r-1),r))\mathrm{d}r$.

Proposition

For each $z>0$, we have

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-L_{z,x}(1,t)}|\tau_0^{t-1}(z,x)>t-1\right]=\mathbb{E}_B^z(e^{-\int_1^\infty\varphi(u(Z(r-1),r))\mathrm{d}r})\in(0,1],$$

where $\{Z(s):s\geq 0\}$ is a Bessel-3 process starting from z under $\mathbb{P}_B^z$.

Proof of Theorem 4

Theorem 4 (Large deviation)

For any $\gamma > \sqrt{2\alpha} - \rho$ and $x > 0$, we have

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(R_t > \gamma t)\frac{\sqrt{2\pi t}}{1-\mathrm{e}^{-2\gamma x}}\mathrm{e}^{(\frac{1}{2}(\gamma+\rho)^2-\alpha)t}\mathrm{e}^{-(\gamma+\rho)x} = C_2,$$

where C_2 is a positive constant independent of x.

Step 1. $\mathbb{P}_{\delta_x}(R_t > \gamma t) = 1 - \exp\{-v^{\gamma t}(x,t)\}$.

Step 2. For any $x, t > 0$, $v^{\gamma t}(x,t) \le \hat{v}_t < \infty$.

Proof of Theorem 4

Step 3. It follows from Feynman-Kac formula that

$$v^{\gamma t}(x,t) = \frac{\mathrm{e}^{\rho x-(\rho^2/2-\alpha)(t-1)}}{\sqrt{2\pi(t-1)}} \int_{(0,\infty)} v^{\gamma t}(z,1)\mathrm{e}^{-\frac{(x-z)^2}{2(t-1)}}\mathrm{e}^{-\rho z} \times \mathbb{E}\left[\mathrm{e}^{-J_{z,x}(0,t-1)}; \tau_0^{t-1}(z,x) > t-1\right]\mathrm{d}z,$$

where $J_{z,x}(0,t-1) = \int_0^{t-1} \varphi(v^{\gamma t}(B_{z,x}^{t-1}(r), r+1))\mathrm{d}r$.

Since the effect of the absorption vanishes as the start position tends to infinity, thus

$$v^{\gamma t}(z+\gamma t,1) \to u^0(z,1) := -\log \mathbb{P}_{\delta_x}(\mathrm{e}^{-\mathcal{X}_t(0,\infty)}), \quad t\to\infty.$$

Proposition

For any $\gamma > \sqrt{2\alpha}-\rho$, $x>0$ and $z\in\mathbb{R}$, we have

$$\lim_{t\to\infty} \mathbb{E}\left[\mathrm{e}^{-J_{z+\gamma t,x}(0,t-1)}; \tau_0^{t-1}(z+\gamma t,x) > t-1\right] = (1-\mathrm{e}^{-2\gamma x})h(z),$$

where $h:\mathbb{R}\to(0,1]$ is a strictly positive function.

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Proposition

For any $\gamma > \sqrt{2\alpha} - \rho$, $x > 0$ and $z \in \mathbb{R}$, we have

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-J_{z+\gamma t,x}(0,t-1)}; \tau_0^{t-1}(z+\gamma t, x) > t-1\right] = (1 - e^{-2\gamma x}) h(z),$$

where $h : \mathbb{R} \to (0,1]$ is a strictly positive function.

Theorem 7 (Tail distribution of the absorbed mass)

When $\rho \geq \sqrt{2\alpha}$, we have that $-\infty < s_0 < 0$ and

$$\mathbb{P}_{\delta_0}\left(\|X_{D_x}^{-\rho}\| > t\right) \sim \frac{u'_{s_0}(x)\mathrm{e}^{-u_{s_0}(x)}}{\mathrm{e}^{-s_0 t}\sqrt{\pi t \psi(s_0)}}, \quad t \to \infty.$$

Recall that

$$u_s(x) := -\log \mathbb{E}_{\delta_0}\left(\mathrm{e}^{-s\|X_{D_x}^{-\rho}\|}\right), \quad x > 0, s \in \mathbb{R}.$$

According to Kyprianou et al. (2012a), when $\rho \geq \sqrt{2\alpha}$, $\{\|X_{D_x}^{-\rho}\| : x \geq 0\}$ is a subcritical CSBP and $u_s(x)$ satisfies the following equation

$$\frac{1}{2}u''_s(x) + \rho u'_s(x) - \psi(u_s(x)) = 0, \quad x > 0,$$

with boundary conditions $u_s(0) = s$ and $u_s(\infty) = 0$.

Based on classical results on some complex differential equations, we analyze the behavior of u_s near its singular point s_0.

Lemma 1

When $\rho \geq \sqrt{2\alpha}$, we have $-\infty < s_0 < 0$, $u_{s_0} > -\infty$, and $u'_{s_0}(0) = 0$.

Lemma 2

When $\rho \geq \sqrt{2\alpha}$, for $x > 0$ fixed, there exists $r_x > 0$ such that $u_s(x)$ is analytic on $V = D(s_0, r_x) \backslash (-\infty, s_0]$ and

$$\partial_s u_s(x) \sim \frac{u'_{s_0}(x)}{2\sqrt{\psi(s_0)(s - s_0)}}, \quad s \to s_0, \ s \in V.$$

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Thanks

Email: zhuyp@pku.edu.cn